

Log-behavior of the Bernoulli Numbers

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Abstract. Let B_n denote the n -th Bernoulli number. Consider the sequence $\{B_{2n}\}_{n \geq 1}$ of nonzero Bernoulli numbers. We prove that $\zeta(x)$ is a log-convex function for $x > 1$, which implies that the sequence $\{|B_{2n}|/(2n!)\}_{n \geq 1}$ is log-convex. As a consequence, we see that the sequence of $\{|B_{2n}|\}_{n \geq 1}$ is log-convex. Moreover, we introduce the function $\theta(x) = (2\zeta(x)\Gamma(x))^{\frac{1}{x}}$, where $\Gamma(x)$ is the Gamma function. We show that $\log \theta(x)$ is strictly increasing for $x \geq 6$. It follows that the sequence of $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$ is strictly increasing. This leads to an affirmative answer to a conjecture of Sun. We further conjecture that $\theta(x)$ is log-convex for $x \geq 6$. If it is true, then it implies the conjecture of Sun concerning the log-convexity of the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$.

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1 Introduction

This note is concerned with the log-behavior of the absolute values of the nonzero Bernoulli numbers. For the background on the Bernoulli numbers, see [3], [5] and [7].

Let B_n denote the n -th Bernoulli number. Recall that $B_{2n+1} = 0$ for $n \geq 1$ and B_{2n} alternate in sign for $n \geq 1$. We consider the sequence $\{|B_{2n}|\}_{n \geq 1}$. The first result of this paper is the log-convexity of the sequence $\{|B_{2n}|/2n!\}_{n \geq 1}$. This implies the log-convexity of the sequence $\{|B_{2n}|\}_{n \geq 1}$. A sequence $\{a_n\}_{n \geq 1}$ of real numbers is said to be log-convex if for any $n \geq 2$,

$$a_n^2 \leq a_{n-1}a_{n+1}.$$

To prove the log-convexity of $\{|B_{2n}|/2n!\}_{n \geq 1}$, we utilize the following relation between the Bernoulli numbers and the Riemann zeta function

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|, \quad (1.1)$$

where

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

By proving that $\zeta(x)$ is log-convex for $x > 1$, we establish the log-convexity of the sequence $\{|B_{2n}|/2n!\}_{n \geq 1}$. Consequently, the sequence $\{|B_{2n}|\}_{n \geq 1}$ is log-convex.

Moreover, we introduce the following function

$$\theta(x) = (2\zeta(x)\Gamma(x))^{\frac{1}{x}}. \quad (1.2)$$

We shall show that $\log \theta(x)$ is strictly increasing for $x \geq 6$. From relation (1.1), it can be checked that

$$\sqrt[n]{|B_{2n}|} = \frac{1}{4\pi^2} \theta^2(2n).$$

The monotone property of $\log \theta(x)$ implies that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$ is strictly increasing. This confirms a conjecture of Sun [8]. We further conjecture that $(\log \theta(x))'' > 0$ for $x \geq 6$. If it is true, then it implies the conjecture of Sun concerning the log-convexity of the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$.

2 The log-convexity of Bernoulli numbers

To prove the log-convexity of Bernoulli numbers, we consider the log-behavior of the Riemann zeta function $\zeta(x)$ for $x > 1$. Recall that a positive function f is called log-convex on a real interval $I = [a, b]$, if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq f(x)^\lambda f(y)^{1-\lambda}, \quad (2.3)$$

see [2]. It is known that a positive function f is log-convex if and only if $(\log f(x))'' \geq 0$. To prove that $\zeta(x)$ is log-convex for $x > 1$, it suffices to show that

$$(\log \zeta(x))'' > 0, \quad (2.4)$$

for $x > 1$.

Lemma 2.1. *The Riemann zeta function $\zeta(x)$ is log-convex for $x > 1$.*

Proof. It is easy to verify that condition (2.4) is equivalent to

$$\zeta(x) \cdot \zeta''(x) - (\zeta'(x))^2 > 0. \quad (2.5)$$

Since $\zeta(x)$ converges for $x > 1$, we find that for $x > 1$,

$$\begin{aligned}
& \zeta(x)\zeta''(x) - (\zeta'(x))^2 \\
&= \sum_{m=1}^{\infty} \frac{1}{m^x} \sum_{n=1}^{\infty} \frac{\log^2 n}{n^x} - \sum_{m=1}^{\infty} \frac{\log m}{m^x} \sum_{n=1}^{\infty} \frac{\log n}{n^x} \\
&= \sum_{n>m\geq 1} \frac{\log^2 n + \log^2 m - 2 \log m \log n}{(mn)^x} \\
&= \sum_{n>m\geq 1} \frac{(\log n - \log m)^2}{(mn)^x},
\end{aligned}$$

which is positive. This completes the proof. \blacksquare

The log-convexity of $\zeta(x)$ enables us to deduce the following log-convex property involving the Bernoulli numbers.

Theorem 2.2. *The sequence $\{\frac{|B_{2n}|}{(2n)!}\}_{n\geq 1}$ is log-convex.*

Proof. Since $\zeta(x)$ is log-convex, setting $x = 2n - 2$, $y = 2n + 2$ and $\lambda = 1/2$ in the defining relation (2.3), we find that

$$\zeta(2n - 2)\zeta(2n + 2) \geq \zeta(2n)^2. \quad (2.6)$$

Invoking the relation (1.1) between $\zeta(x)$ and B_n , we obtain that

$$\left(\frac{|B_{2n}|}{2n!}\right)^2 \leq \frac{|B_{2n-2}|}{(2n-2)!} \cdot \frac{|B_{2n+2}|}{(2n+2)!}.$$

This completes the proof. \blacksquare

Since $(2n!)^2 < (2n-2)! \cdot (2n+2)!$ for $n \geq 1$, it is easy to see that the above theorem implies the log-convexity of the sequence $\{|B_{2n}|\}_{n\geq 1}$.

Corollary 2.3. *The sequence $\{|B_{2n}|\}_{n\geq 1}$ is log-convex.*

3 Log-behavior of $\theta(x)$

Now let us consider the log-behavior of the function

$$\theta(x) = (2\zeta(x)\Gamma(x))^{\frac{1}{x}}.$$

We begin with the following monotone property of $\log \theta(x)$.

Theorem 3.1. *For $x \geq 6$, $\log \theta(x)$ is strictly increasing.*

Proof. To prove $\log \theta(x)$ is increasing for $x \geq 6$, we aim to show that

$$(\log \theta(x))' > 0, \quad (3.7)$$

for $x \geq 6$. Let $g(x) = 2\zeta(x)\Gamma(x)$. We have

$$\begin{aligned} (\log \theta(x))' &= \left(\frac{1}{x} \log(2\zeta(x)\Gamma(x)) \right)' = \left(\frac{1}{x} \log g(x) \right)' \\ &= -\frac{1}{x^2} \log g(x) + \frac{1}{x} \frac{g'(x)}{g(x)} \\ &= \frac{1}{x} \left(\frac{g'(x)}{g(x)} - \frac{\log g(x)}{x} \right). \end{aligned}$$

Thus (3.7) can be rewritten as

$$\frac{g'(x)}{g(x)} > \frac{\log g(x)}{x},$$

for $x \geq 6$. Note that $g(x)$ is continuous and differentiable on $(1, \infty)$, since $\zeta(x)$ and $\Gamma(x)$ are continuous and differential on $(1, \infty)$. Therefore, one can apply the mean value theorem on $[2, x]$ to estimate $\log g(x)/x$. We claim that there exists t in $(2, x)$ such that

$$\frac{g(t)'}{g(t)} > \frac{\log g(x)}{x}. \quad (3.8)$$

If the above inequality holds, then it suffices to show that

$$\frac{g(x)'}{g(x)} > \frac{g(t)'}{g(t)}. \quad (3.9)$$

We now proceed to prove (3.8). Since $\zeta(2) = \frac{\pi^2}{6}$ and $\Gamma(2) = 2$, we find that

$$\log g(2) = \log(2\zeta(2)\Gamma(2)) = \log \frac{2\pi^2}{3} < 2. \quad (3.10)$$

On the other hand, for $x \geq 6$, we have $\zeta(x) > 1$ and $\Gamma(x) > e^x$. It follows that

$$\log g(x) = \log 2 + \log \zeta(x) + \log \Gamma(x) > x. \quad (3.11)$$

In the view of (3.10) and (3.11), we deduce that for $x \geq 6$,

$$\frac{\log g(x)}{x} = \frac{(1 - 2/x) \log g(x)}{(1 - 2/x)x} < \frac{\log g(x) - 2}{x - 2} < \frac{\log g(x) - \log g(2)}{x - 2}. \quad (3.12)$$

Applying the mean value theorem to $\log g(x)$, we see that there exists $t \in (2, x)$ such that

$$(\log g(t))' = \frac{\log g(x) - \log g(2)}{x - 2}, \quad (3.13)$$

that is,

$$\frac{g'(t)}{g(t)} = \frac{\log g(x) - \log g(2)}{x - 2}. \quad (3.14)$$

Combining (3.12) and (3.14), we get (3.8).

It remains to prove (3.9). We claim that for $y > 1$,

$$\left(\frac{g'(y)}{g(y)} \right)' > 0. \quad (3.15)$$

Since

$$\left(\frac{g'(y)}{g(y)} \right)' = (\log g(y))'' = (\log \Gamma(y))'' + (\log \zeta(x))'',$$

we see that (3.15) holds as long as we can show that $(\log \Gamma(y))'' > 0$ and $(\log \zeta(x))'' > 0$ for $y > 1$. It is a known fact that $(\log \Gamma(y))'' > 0$ for $y > 1$, see Andrews, Askey and Roy [1, Theorem. 1.2.5]. On the other hand, in the proof of Lemma 2.1, we have shown that $(\log \zeta(x))'' > 0$. This proves (3.15). It follows that $\frac{g'(y)}{g(y)}$ is strictly increasing for $y > 1$. Thus for $2 < t < x$, we have inequality (3.9).

Combining (3.8) and (3.9), we deduce that for $x \geq 6$,

$$\frac{g'(x)}{g(x)} - \frac{\log g(x)}{x} > \frac{g'(x)}{g(x)} - \frac{g'(t)}{g(t)} > 0.$$

It follows that $(\log \theta(x))' > 0$ for $x \geq 6$. This completes the proof. \blacksquare

From the log-behavior of $\theta(x)$, we are led to an affirmative answer to a conjecture of Sun [8].

Corollary 3.2. *The sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$ is strictly increasing.*

Proof. From the relation (1.1), we see that for $n \geq 1$

$$\sqrt[n]{|B_{2n}|} = \frac{1}{4\pi^2} \sqrt[n]{2\zeta(2n)(2n)!} = \frac{1}{4\pi^2} \theta^2(2n). \quad (3.16)$$

Since $\log \theta(x)$ is strictly increasing for $x \geq 6$, we see that $\theta(x)$ is also strictly increasing for $x \geq 6$. It follows from (3.16) that $\sqrt[n]{|B_{2n}|}$ is strictly increasing for $n \geq 3$. On the other hand, it is easily checked that

$$|B_2| < \sqrt{|B_4|} < \sqrt[3]{|B_6|}.$$

This completes the proof. \blacksquare

In closing, we pose the following conjecture. If it is true, then it implies the log-convexity of the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$ as conjectured by Sun [8].

Conjecture 3.3. *The function $\theta(x) = (2\zeta(x)\Gamma(x))^{\frac{1}{x}}$ is log-convex, that is, $(\log f(x))'' > 0$ for $x \geq 6$.*

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